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Discretization-induced delays and their role in the dynamics

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Abstract
We show that a discretization of a continuous system may entail 'hidden' delays and thus introduce instabilities. In this case, while the continuous system has an attractive fixed point, the instabilities present in the equivalent discrete one may lead to the appearance of a limit cycle. We explain that it is possible, thanks to the proper staggering of the discrete variables, to eliminate the hidden delay. However, in general, other instabilities may appear in the discrete system which can even lead to chaotic behaviour.

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1. Introduction

Discretization is a delicate procedure. While taking the continuous limit of a discrete system is (usually) a straightforward procedure, the converse operation is highly non-trivial. There is no limit to the number of discrete analogues to a given continuous (differential) equation one can find (admittedly, with a dose of imagination). So the question arises, 'is there a way to define the *proper* discretization of a given differential system?' An answer to this question can be given in a very general, albeit abstract, way. If a discretization preserves most properties of the continuous system, or in any case, the ones we deem essential, then it is a good discretization.

Let us present a well-known example of how things can go wrong. We start with the first-order differential equation

$$X' = \alpha X - \beta X^2, \tag{1.1}$$

introduce a time step ϵ and discretize the derivative as $X' = (X_{n+1} - X_n)/\epsilon$. Rescaling X as $x = \epsilon\beta X/a$ we obtain as a possible discretization

$$x_{n+1} = a x_n (1 - x_n) \tag{1.2}$$

where $a = 1 + \epsilon\alpha$.

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Equation (1.2) is of course a discretization of (1.1) in the sense that it contains it as a continuous limit. However this connection is valid only for very small values of the time step ϵ. If we put ourselves in a purely discrete setting with ϵ of order unity then equation (1.2) has nothing in common with (1.1). While the latter is reduced to a linear equation through the transformation $X = 1/Y$, the former, which is usually referred to as the logistic mapping [1], is well known for the chaotic behaviour of its solutions (at least for a range of its parameter). Is there a better discretization of (1.1)? The answer is (obviously) 'yes'. If we require that the resulting discrete system be linearizable through the same transformation as above, $x_n = 1/y_n$, then there exists a unique discretization [2] in the form of a homographic mapping

$$x_{n+1} = \frac{ax_n}{1 + x_n} \tag{1.3}$$

with the same definition for x and a as in equation (1.2).

This is a particular instance of the strength of the integrability argument. Equation (1.1), obviously a Riccati equation, is one of the simplest integrable equations. By requiring that its discretization preserve integrability, we end up with a unique solution: equation (1.3). As we explained in [3], the integrability requirement is sufficient in order to isolate one, or, sometimes, a few *proper* discretizations among a slew of possible ones.

While the integrability argument is most useful, it does not help at all when the system is not integrable. In this case, the only possible approach is to try to preserve the essential dynamical behaviour of the continuous system (and moreover to do this for as large a domain of the discretization parameter as possible) [4]. This is a tall order indeed, because there exist no criteria of 'preservation of the essential dynamical behaviour' (due, of course, to the vagueness of this statement), contrary to the integrability situation where efficient criteria do exist. Still in specific cases one can, perhaps with some effort, obtain satisfactory answers.

In this paper we shall examine a system the dynamics which have been well studied. Depending on one parameter the orbits of the system are either attracted to a fixed point or to a limit cycle (and no other dynamical behaviour was found). We present a discretization of this system and show how the choice of the discretization can influence the dynamics. In particular, we show that too naïve a discretization may comprise 'hidden' delays which can alter the dynamics of the system.

2. A differential-delay system

The system we intend to analyse in this paper is a differential-delay one introduced in [5] as a possible model for gene dynamics. It has the form

$$x' = 1 - xy \qquad y' = f(\tilde{x}) - y \tag{2.1}$$

where the tilde in the argument of f indicates that the dependent variable is taken at time $t - \tau$. We have put to unity three coefficients with appropriate scalings of the independent and dependent variables, which we take to be positive: $x(t), y(t) > 0$. The function $f(x)$ was taken in the form of a steep 'sigmoid' in [5] but since this choice was directed by the particular biological setting, in what follows we shall make a simpler assumption $f(x) = \alpha x^\nu$ $(\alpha, \nu > 0)$.

Equation (2.1) has a fixed point given by the equations

$$x_0 y_0 = 1 \qquad y_0 = f(x_0). \tag{2.2}$$

In order to study the stability of the fixed point, we linearize by putting $x = x_0 + \xi$, $y = y_0 + \eta$ for small ξ, η. We obtain the system

$$\xi' = -y_0\xi - x_0\eta, \qquad \eta' = f'(x_0)\tilde{\xi} - \eta. \tag{2.3}$$

Next we seek a solution of (2.3) proportional to $e^{\rho t}$ and obtain for ρ the equation

$$\begin{vmatrix} y_0 + \rho & x_0 \\ -f'(x_0)\,e^{-\rho\tau} & 1 + \rho \end{vmatrix} = 0. \tag{2.4}$$

Using the fact that $f(x) = \alpha x^\nu$ we have $x_0 f'(x_0) = \nu f(x_0)$ and using the fixed-point equations we finally have

$$(\rho + 1)(\rho + y_0) + \nu y_0\,e^{-\rho\tau} = 0. \tag{2.5}$$

We look for solutions with $\mathrm{Re}(\rho) > 0$. From (2.5) it is clear that in the absence of delay there exists no possibility for instability. In order to study (2.5) in the presence of delay, we consider a complex eigenvalue $\rho = \sigma + i\omega$. At the threshold of instability we have $\sigma = 0$. Separating the real and imaginary parts of (2.5) we find at the threshold:

$$y_0 - \omega^2 + \nu y_0 \cos\omega\tau = 0, \qquad (1 + y_0)\omega - \nu y_0 \sin\omega\tau = 0. \tag{2.6}$$

Eliminating the sine and cosine we obtain an equation for ω:

$$\omega^4 + \left(1 + y_0^2\right)\omega^2 + y_0^2(1 - \nu^2) = 0. \tag{2.7}$$

Clearly, a real, positive root for ω^2 can exist only provided $\nu > 1$.

When ν is close to one, we can obtain an estimate of the delay τ from (2.6) and (2.7). We find that $\tau = \pi/\omega$ with $\omega^2 = 2y_0^2(\nu - 1)\big/\left(1 + y_0^2\right)$, i.e., the minimal value of the delay necessary for the appearance of an instability diverges as $\nu \to 1^+$. On the other hand, when the nonlinearity becomes very large, i.e. for $\nu \gg 1$, one can easily obtain the asymptotic dependence of this critical delay in terms of the exponent ν and the fixed point y_0:

$$\tau \sim \frac{1 + y_0}{y_0}\frac{1}{\nu}. \tag{2.8}$$

Of course, physically, for a fixed parameter α in (2.1), $y_0 \to 1$ as $\nu \to \infty$ and hence $\tau \sim 2/\nu$.

It is also easily shown that once a delay τ exceeds the critical value derived from (2.6), the ensuing instability will always be present. More precisely, as functions of τ, the solutions $\sigma(\tau)$ and $\omega(\tau)$ of (2.5) are always differentiable at a critical point $\sigma(\tau) = 0$ and the gradient of $\sigma(\tau)$ at such a point is always strictly positive. (Hence for increasing τ, once a delay-induced instability appears, it necessarily persists.)

Solving numerically (2.5) we can obtain the instability domain in the ν, τ plane. The results are shown in figure 1.

Choosing parameters inside the instability domain leads to a solution of (2.1) exhibiting asymptotically constant amplitude oscillations: the solution is attracted to a limit cycle [6]. In figure 2 we present two different trajectories starting from the same initial condition for two different values of the delay, one leading to a fixed point (dashed line) and the other to a limit cycle (continuous line).

3. Discretizing the differential delay system

Given a differential system there exist an infinite number of possible discrete forms. In order to discretize system (2.1) we shall consider a few common scenarios with the only constraint that the final system involves only positive terms. (This is a standard procedure [7] in order to produce ultradiscretizable systems, i.e., systems which give rise to generalized cellular automata [8].) We introduce a time step ϵ and write the derivative as $x' = (x_{n+1} - x_n)/\epsilon$. One

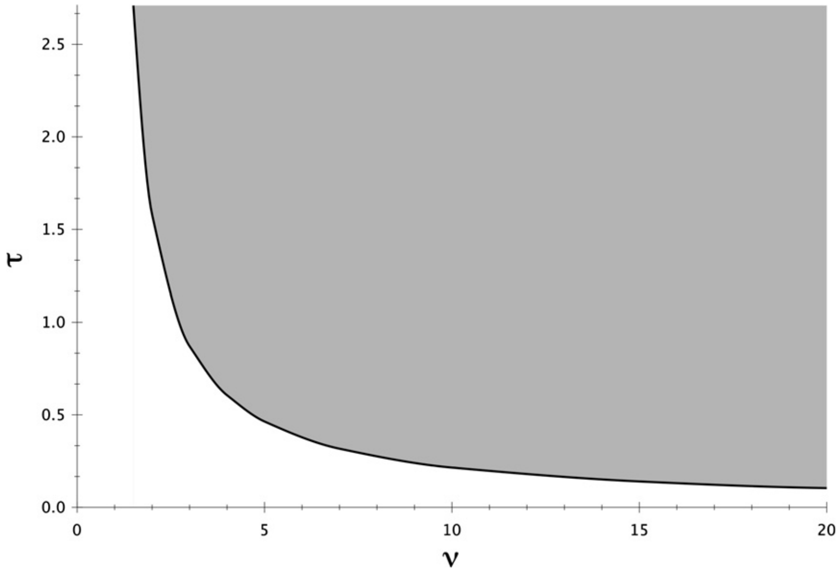

Figure 1. Plot of the threshold delay τ above which instability sets in for system (2.1) (grey zone), in terms of the exponent ν. The equilibrium value y_0 was set equal to 1.

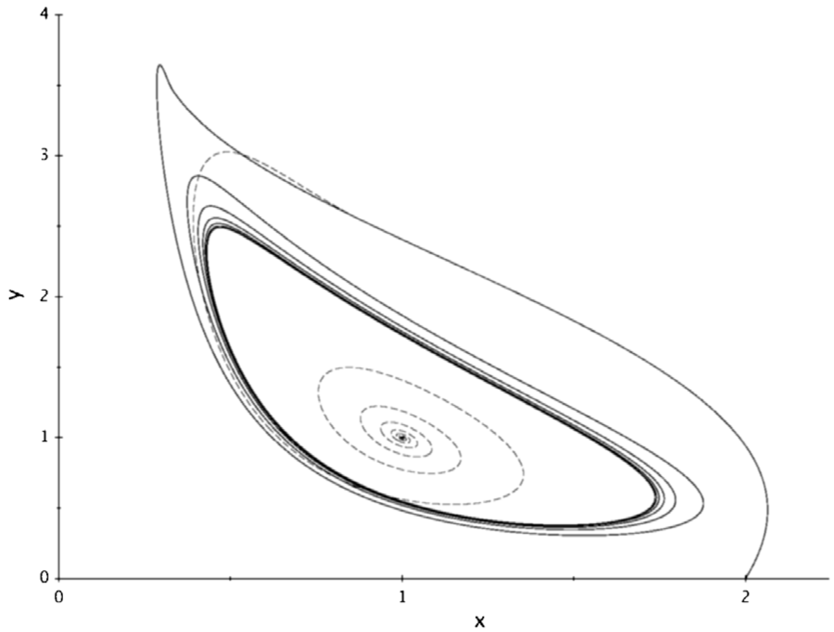

Figure 2. Two typical trajectories for the system (2.1) with exponent $\nu = 2$ and equilibrium $y_0 = 1$ (the critical delay τ at which instability sets in is $\pi/2$ in this case). Both trajectories were calculated for the same initial point (2, 0): the dashed trajectory for a subcritical delay $\tau = 1$, the other for $\tau = 2$.

way to guarantee the positivity is to write the xy term in the first equation of (2.1) as $x_{n+1}y_n$ and move it to the left-hand side. Rescaling as $x = \epsilon X$, $y = Y/\epsilon$, we obtain the discrete system

$$X_{n+1} = \frac{1+X_n}{1+Y_n}, \qquad Y_{n+1} = f(X_{n-k}) + \lambda Y_n \tag{3.1}$$

where $\lambda = 1-\epsilon$ $(0 < \lambda < 1)$. The index $n-k$ in the argument of the function $f(x)$ indicates the presence of a delay. The function $f(x)$ is taken, in analogy with the continuous case, as $f(X) = \beta X^\nu$, with $\beta = \alpha\epsilon^{\nu+2}$ (and thus we have $Xf'(X) = \nu f(X)$.)

As we have seen in the analysis of system (2.1) the existence of a delay is essential for the appearance of instability. So the question naturally arises: is this still true for the discrete system (3.1)? In order to investigate this we start with the system

$$X_{n+1} = \frac{1+X_n}{1+Y_n}, \qquad Y_{n+1} = f(X_n) + \lambda Y_n. \tag{3.2}$$

The unique fixed point of (3.2) is given by the equation $X_0Y_0 = 1$, $(1-\lambda)Y_0 = f(X_0)$. The stability analysis of (3.2) is quite straightforward. We linearize around the fixed point, put $\theta = 1/(1+Y_0)$ and seek a solution proportional to ρ^n. We find that ρ is given by the eigenvalue problem

$$\begin{vmatrix} \theta-\rho & -\theta/Y_0 \\ f'(X_0) & \lambda-\rho \end{vmatrix} = 0 \tag{3.3}$$

or equivalently

$$\rho^2 - (\lambda+\theta)\rho + \lambda\theta + \nu(1-\lambda)(1-\theta) = 0. \tag{3.4}$$

The condition for the existence of instability is that the modulus of ρ be larger than 1. Given the form of (3.4) the only possibility for instability is that the two roots of (3.4) be complex conjugate. (Indeed, the position of the minimum of the lhs of (3.4) lies between 0 and 1 and the value of the trinomial is positive at 0 and 1. Moreover the roots, if they were real, would have the same sign. Thus they should lie between 0 and 1, which is not what is sought.)

The condition for the roots to be complex conjugate is

$$(\lambda-\theta)^2 - 4\nu(1-\lambda)(1-\theta) < 0 \tag{3.5}$$

while to have $|\rho|^2 > 1$ we have

$$\lambda\theta + \nu(1-\lambda)(1-\theta) > 1. \tag{3.6}$$

Since $\lambda < 1, \theta < 1$, it is easy to check that if (3.6) is satisfied then (3.5) is automatically true. Thus if

$$\nu > \frac{1-\lambda\theta}{(1-\lambda)(1-\theta)} = \frac{1}{1-\lambda} + \frac{1}{1-\theta} - 1 = \frac{1}{1-\lambda} + \frac{1}{Y_0} \equiv \nu_A \tag{3.7}$$

the fixed point of (3.2) is unstable. Thus it is always possible, given the parameter λ and the fixed point Y_0, to choose ν so as to lie in the unstable domain. Taking for instance $\lambda = 1/2$, $Y_0 = 1$, we can predict that for $\nu > 3$ the dynamics of the system will be attracted to a limit cycle. (As we shall see in numerical examples presented in the following section this is indeed the case.)

This result may look astonishing in view of our findings on the continuous system where a delay is essential for instability. The explanation, however, is simple: the 'naïve' discretization (3.2) entails a *hidden* delay. The first equation of the system comprises the variables X_n, X_{n+1} and Y_n which belong to the sequence $\ldots, Y_{n-1}, X_n, Y_n, X_{n+1}, Y_{n+1}, \ldots$. Since these three contiguous variables appear in the first sequence it would be natural for the second equation to relate to the variables, Y_n, X_{n+1} and Y_{n+1}. The fact that X_n appears instead of X_{n+1} means that this variable is delayed by one time step. Thus, the naïve discretization (3.2) comprises a hidden delay which would explain the appearance of instability.

A natural question thus arises: what would happen if the discretization were performed in the form of a staggered [9] system? We consider the system

$$X_{n+1} = \frac{1 + X_n}{1 + Y_n} \qquad Y_{n+1} = f(X_{n+1}) + \lambda Y_n \tag{3.8}$$

which has the same fixed point as (3.2). Its stability analysis leads to the equation

$$\rho^2 - (\lambda + \theta - \nu(1 - \lambda)(1 - \theta))\rho + \lambda\theta = 0. \tag{3.9}$$

Clearly we can never have two complex conjugate roots with $|\rho|^2 > 1$. Thus the instability which was present in equation (3.2) has disappeared: the proper staggering of variables in (3.8) has done away with the hidden delay.

Does the conclusion of the previous paragraph guarantee the nice behaviour of the solutions of (3.8) for any ν? This turns out not to be the case. From (3.9) it is clear that the two roots have the same sign and that their product is smaller than 1. Since the value of the lhs of (3.9) at 0 and 1 is larger than 0 the two roots, if they were positive, would have moduli smaller than 1 which is not what is sought. However it is possible for *one* root to have modulus larger than 1 if both roots are negative. Computing the value of the left-hand side of (3.9) at $\rho = -1$ we find that if

$$\nu > \frac{(1 + \lambda)(1 + \theta)}{(1 - \lambda)(1 - \theta)} \equiv \nu_B = \nu_A + \frac{\lambda + \theta + 2\lambda\theta}{(1 - \lambda)(1 - \theta)} \tag{3.10}$$

there exists a possibility of having one real root with $|\rho| > 1$. For the typical values $\lambda = 1/2, Y_0 = 1$, this happens for $\nu > 9$. A careful study of the dynamics of (3.8) in a domain where (3.10) is satisfied reveals a possibility for chaotic behaviour. In the following section we shall present numerical results which will illustrate the above conclusions.

Summarizing our findings up to now we have two discretizations of equation (1.1). The first one, being of the form $X_{n+1} = g(X_n, Y_n), Y_{n+1} = h(X_n, Y_n)$ (which is the customary way of presenting a two-component discrete system), leads to a certain instability of the fixed point. The second, which is based on the correct staggering of variables, $X_{n+1} = g(X_n, Y_n), Y_{n+1} = h(X_{n+1}, Y_n)$ does not present such an instability. However, another kind of instability does exist in this second case, although it appears for larger values of the exponent ν than that of the first case. (Since X and Y do not play the same role one could wonder whether the other possible staggering, $\dots, X_{n-1}, Y_n, X_n, Y_{n+1}, X_{n+1}, \dots$, would lead to a different conclusion. It turns out that this is not the case as this staggering leads to a system related to (3.8) through a simple variable transformation.)

4. A different discretization

In the previous section we have analysed two discretization schemes which constitute two ‘extreme’ situations: staggered and non-staggered. However it is possible by combining these two discretizations to devise yet another discretization scheme. We introduce the system:

$$X_{n+1} = \frac{1 + X_n}{1 + Y_n}, \qquad Y_{n+1} = \phi(X_n, X_{n+1}) + \lambda Y_n \tag{4.1}$$

where $\phi(X_n, X_{n+1}) = \gamma X_n^\kappa X_{n+1}^\mu$ with $\kappa + \mu = \nu$ ($\kappa, \mu > 0$), i.e., only part of the argument of the function ϕ is staggered.

In order to study the stability of the fixed point, we linearize around it and seek a solution proportional to ρ^n. After some straightforward calculations we obtain for ρ the equation:

$$\rho^2 - (\lambda + \theta - M)\rho + \lambda\theta + K = 0 \tag{4.2}$$

where $M = \mu(1 - \lambda)(1 - \theta)$, $K = \kappa(1 - \lambda)(1 - \theta)$.

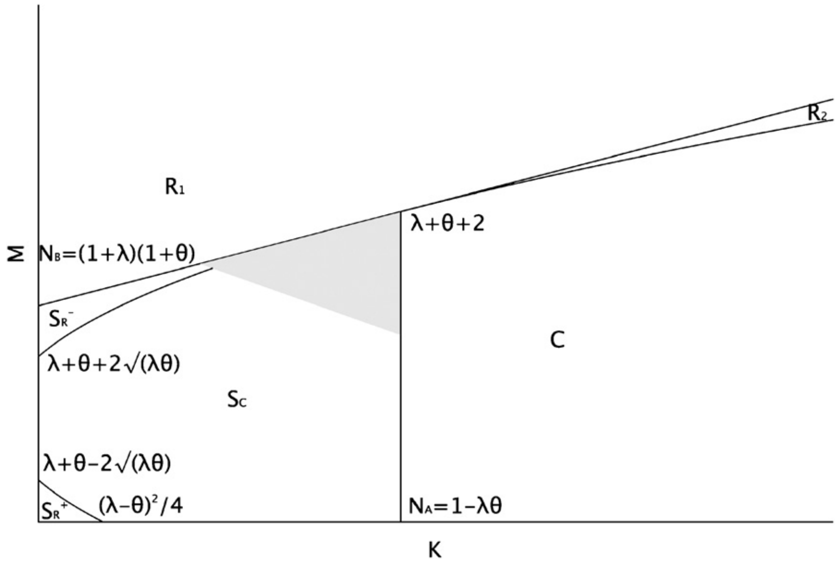

Figure 3. The stability/instability regions in the *K–M* plane for system (4.1), as analysed in section 4.

In order to study the possibilities for instability of the fixed point we start by remarking that the two roots of (4.2) have the same sign and that the lhs of (4.2) is positive at $\rho = 0$ and $\rho = 1$. A quantity that is useful in our analysis is the position of the minimum of the trinomial on the lhs of (4.2): $\rho_m = (\lambda + \theta - M)/2$. We remark that it is always true that $\rho_m < 1$. First we consider the case where equation (4.2) has complex roots. The condition for this is that the discriminant be $\Delta = (\lambda + \theta - M)^2 - 4(\lambda\theta + K) < 0$. In this case, if for the product of the roots we have $\lambda\theta + K < 1$, the fixed point is stable (S_C). In the opposite case $\lambda\theta + K > 1$ the fixed point is unstable (C). The case of real roots corresponds to a positive discriminant: $\Delta = (\lambda + \theta - M)^2 - 4(\lambda\theta + K) > 0$. If $\rho_m > 0$ the two roots are positive and they should be smaller than 1. This corresponds to a stable fixed point $\left(S_R^+\right)$. We turn now to the case $\rho_m < 0$ and we examine the value of the lhs of (4.2) at $\rho = -1$. If we have $1 + (\lambda + \theta - M) + \lambda\theta + K < 0$ then the roots are negative with one of them lying in the interval $[0, -1]$ and one larger in absolute value than 1. This case corresponds to an unstable fixed point (R_1). If on the other hand we have $1 + (\lambda + \theta - M) + \lambda\theta + K > 0$ then both roots must be either larger or smaller than -1. In the first case their product must be $\lambda\theta + K < 1$ and the fixed point is stable (S_R^-) while in the latter case, $\lambda\theta + K > 1$, the fixed point is unstable (R_2). The situation is summarized in figure 3 where we have drawn the boundaries between the various regions of (in)stability in the *K–M* plane. (In the figure we are using the obvious notations $N_A = \nu_A(1 - \lambda)(1 - \theta)$ and $N_B = \nu_B(1 - \lambda)(1 - \theta)$.)

The interest in the discretization (4.1) is that it includes both (3.2) and (3.8) introduced in section 3 and could thus provide a discretization slightly better, from the stability point of view, than these two. This is indeed the case since the stability region is apparently more extended than one would have predicted from the values of ν_A and ν_B. However a detailed (numerical) study of the behaviour of the system reveals the existence of a domain (in light grey in figure 3) of *nonlinear* instability which considerably reduces the size of the domain of stability.

Let us comment here on the appearance of the orbits in the various domains of the *K–M* plane and give some numerical examples of the possible dynamical situations. We start from

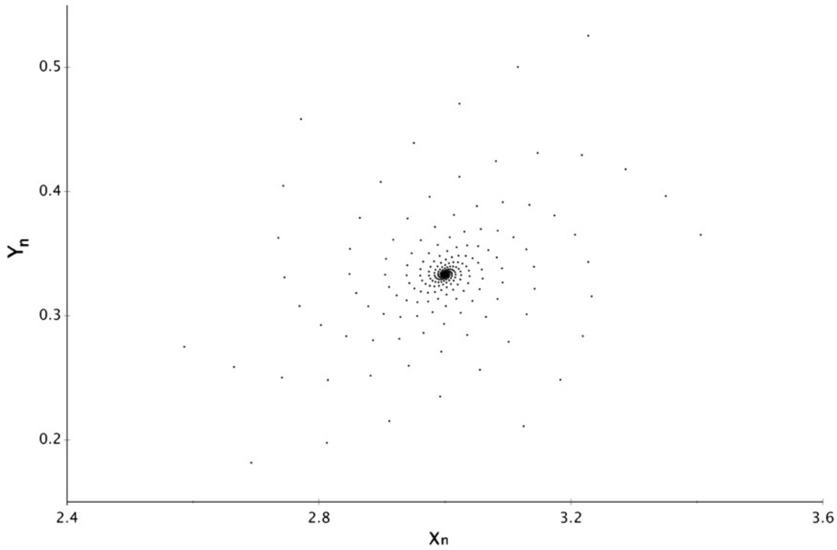

Figure 4. Plot of a trajectory attracted to a fixed point obtained for (4.1) with exponents $\mu = 0, \kappa = 4$ and parameters $\lambda = 0.15, \theta = 0.75$ ($\nu_A = 71/17 \simeq 4.176\ldots$). The initial point was taken at (2.5, 0.3).

the $M = 0$ axis. For values of K below $(\lambda - \theta)^2/4$, i.e. in the region S_R^+, the initial conditions are directly attracted to the fixed point without spiraling. When we choose a value of K in the S_C region the initial conditions still converge to the fixed point but in this case along a spiraling trajectory. In figure 4 we show such an example for a value of K slightly smaller than the threshold $N_A = 1 - \lambda\theta$. When the value of K increases beyond this threshold the instability leads first to the appearance of a limit cycle, as shown in figure 5. For still larger values of K the limit cycle gets deformed, the orbit 'slows down' around some points, figure 6, and eventually breaks up completely into a simple periodic orbit with a small number of points. Increasing further the value of K we enter a chaotic regime.

On the $K = 0$ axis now, we have two stable regions already explored, namely S_R^+ and S_C. When the value of M becomes larger than $\lambda + \theta + 2\sqrt{\lambda\theta}$ we enter the stability domain S_R^-. Here the roots of (4.2) are negative and the convergence to the fixed point proceeds through oscillations of decreasing amplitude (rather than rotations as in the case of spiraling). When the value of M reaches the threshold $N_B = (1 + \lambda)(1 + \theta)$ the amplitude of the oscillations does no longer decrease and we have the appearance of a period-two periodic point. When we increase still further the value of M we observe a period doubling but at the same time the attraction basin of the 2^n-periodic points shrinks and a triply periodic point also appears. The latter also follows a period doubling route. In figure 7 we show such a situation where we have a 3×2^2 period coexisting with a 2^4 one.

For increasing M, the two families of periodic orbits progressively merge into a single chaotic attractor. An example of the latter is shown in figure 8. Working with finite values of both K and M does not lead to significantly different dynamical behaviour from the one observed along the axes in each of the stability (or instability) regions. Moreover no new phenomena appear in the instability region R_2. The most interesting situation is the one encountered in the upper region of S_C. While the fixed point is linearly stable, its attraction basin progressively shrinks and *nonlinear* instabilities set in. Again we encounter triangle-shaped orbits, i.e. triply periodic points, which follow a period doubling route to chaos.

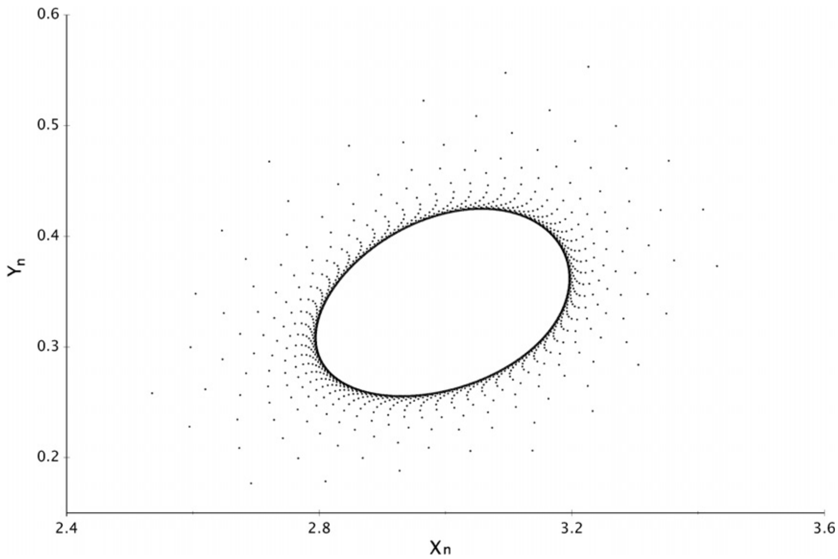

Figure 5. Plot of a trajectory attracted to a limit cycle obtained for (4.1) with exponents $\mu = 0, \kappa = 4.2$ and parameters $\lambda = 0.15, \theta = 0.75$ ($\nu_A = 71/17 \simeq 4.176\ldots$). The initial point was taken at (2.5, 0.3).

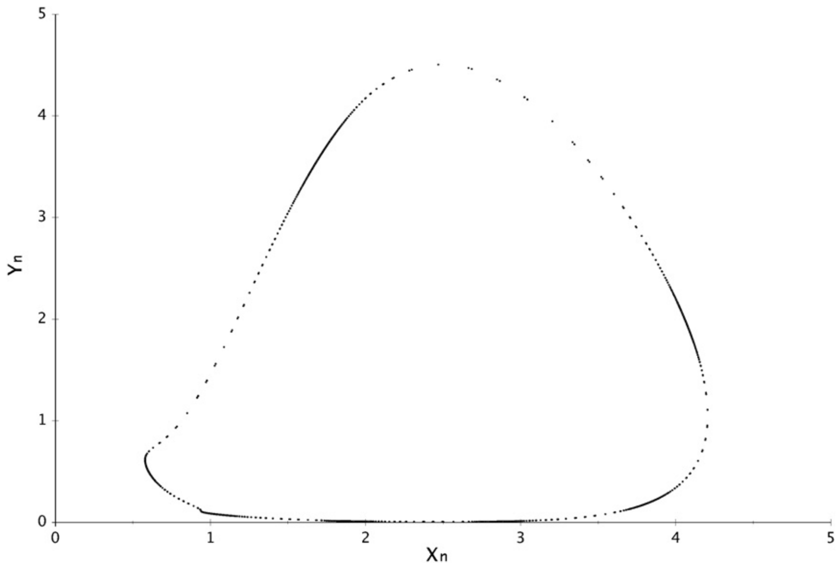

Figure 6. Plot of a limit cycle near a periodic orbit, obtained for system (4.1) with exponents $\mu = 0, \kappa = 8.08$ and parameters $\lambda = 0.15, \theta = 0.75$ ($\nu_A = 71/17 \simeq 4.176\ldots$). The initial point was taken at (2.5, 0.3).

Periodic orbits with more points also exist and as a consequence the whole greyed region in figure 3 is a region of instability (except in the very small basin of attraction around the stable fixed point).

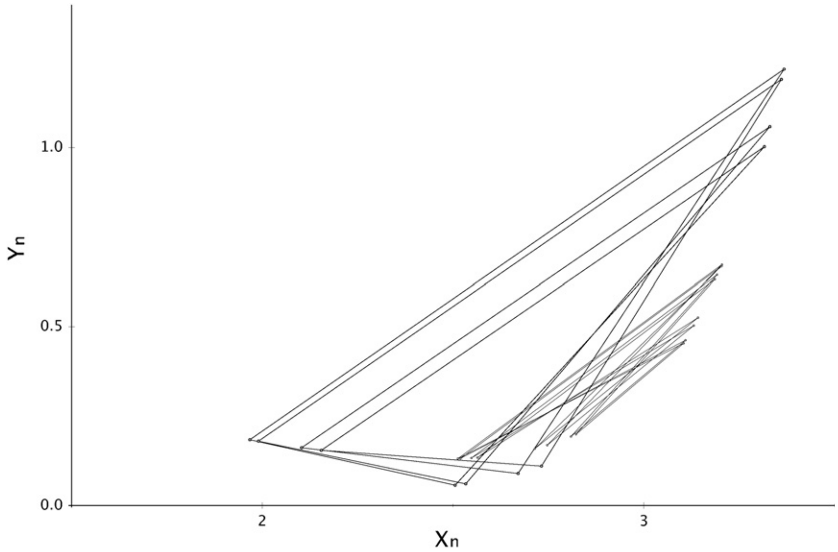

Figure 7. Plot of a period 3×2^2 and period 2^4 orbit, both obtained for system (4.1) with exponents $\mu = 12.55$, $\kappa = 0$ and parameters $\lambda = 0.15$, $\theta = 0.75$ ($\nu_B = 161/17 \simeq 9.47\ldots$). The respective initial values were (2.5, 0.3) for the 3×2^2 orbit and (0.5, 0.3) for the 2^4 orbit.

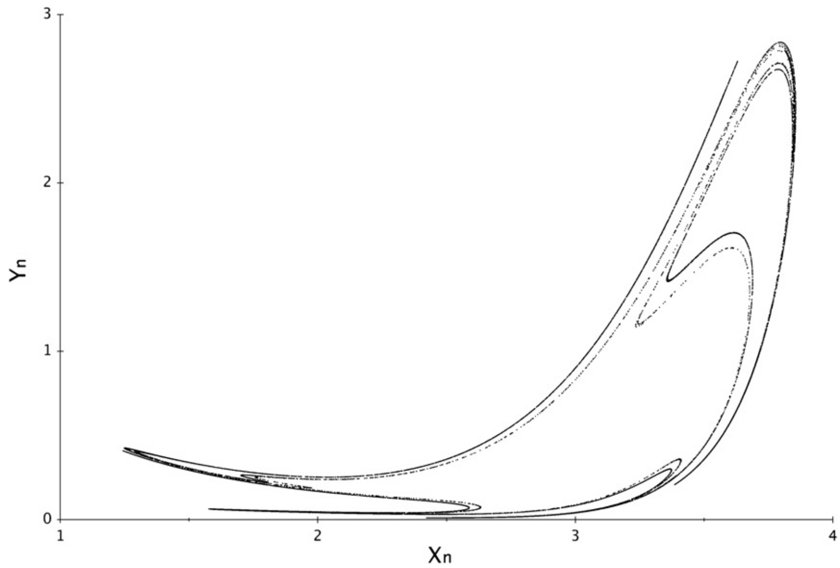

Figure 8. Plot of a chaotic attractor obtained for system (4.1) with exponents $\mu = 8$, $\kappa = 6$ and parameters $\lambda = 0.15$, $\theta = 0.75$ ($\nu_B = 161/17 \simeq 9.47\ldots$, $\nu_B + \nu_A = 232/17 \simeq 13.647\ldots$, i.e. well within the unstable C region in the κ–μ plane). The initial point was taken at (0.5, 0.3).

5. Conclusion

In this paper we have analysed the difficulties associated with the discretization of a continuous (differential) system, illustrating them through a specific model. We have argued that, depending on the mode of discretization used, one may introduce instabilities which are not present in the continuous model. In particular, we have shown that the discretization of a purely differential system (without delay) can artificially introduce delays which may alter the dynamical behaviour of the system.

In order to visualize the 'hidden' delay present in equation (3.2) and absent in (3.8), we can consider the following asymptotic argument. We saw in section 2 that for large ν the condition for instability between τ and ν is of the form $\tau > \frac{y_0+1}{y_0}\frac{1}{\nu}$ (2.8), where y_0 is the value of y at the fixed point, or, equivalently, that for the critical values of τ we must have $\tau\nu = \frac{y_0+1}{y_0}$. We now consider the continuous limit of ν_A (3.7) and find $\nu_A = \frac{y_0+1}{y_0}\frac{1}{\epsilon}$. At this limit, for a delay time we must have $\tau = k\epsilon$, given the discretization we introduced in section 2. We thus obtain $\tau\nu_A = \frac{y_0+1}{y_0}k$ which, compared to the value of $\tau\nu$ obtained above, implies that we have $k = 1$. Thus the 'hidden' delay in (3.2) is exactly one time step, as suggested by the staggering argument used in section 3. The discretization (3.8), on the other hand, leads to a limit $\nu_B \approx \frac{4}{y_0}\frac{1}{\epsilon^2}$ at dominant order. Thus we have $\tau\nu_B = \frac{4}{y_0}\frac{k}{\epsilon}$ and the 'delay' k should vanish like ϵ. This means that the discretization (3.8), which was staggered compared to (3.2), does not entail hidden delays.

Still, as shown in the previous sections, even discretizations which do not introduce delays may have other instabilities which, in some cases, can even lead to chaotic behaviour. Clearly when one is faced with the delicate task of discretization one needs some guide. In our previous works we have followed the rule that a discretization must preserve the properties and behaviour of the continuous system, but, as the present analysis shows, this is not always possible. The only case (to our knowledge) where the discrete form of some continuous system is precisely fixed, is when one discretizes an integrable continuous system while preserving the integrable character. The constraints of integrability are such that the integrable discrete analogue is unambiguously defined. Unfortunately, integrable systems are rare and thus in most (nonintegrable) cases one does not have a clear guide as to what is the optimal discrete form. The present paper may serve as another *caveat* that particular care is always necessary when one tackles the problem of discretization.

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